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Modules over Affine Algebras Having Subexponential Growth

LOUIS ROWEN

Bar-Ilan University, Ramat-Gan, Israel

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Aljadeff and Rosset [1] showed that free modules over affine algebras of subexponential growth satisfy the "unique rank" property, otherwise known as IBN (invariant base number). The object of this note is to modify their proof to obtain a rank on all finitely generated (f.g.) modules over an affine algebra R over a field F, which for f.g. free modules coincides with the customary rank of a free module when R has subexponential growth. At times this turns out to be a Sylvester rank function, in the sense of Schofield [4].

Suppose $S=\{r_1, ..., r_k\} \subset R$ and $R=F\{S\}=F\{r_1, ..., r_k\}$. Let $R_n = \sum_{j=0}^{n} FS^j$ and $d_n=[R_n:F]$. R has *subexponential growth* if $\overline{\lim}\,(\log(d_n)/n) = 0$. By "log" we *always* mean natural logarithm, i.e., to the base e. Also our limits are always taken as $n \to \infty$.

Remark 0. $d_{n+1} \leqslant (|S|+1)\, d_n$, so $d_{n+c}/d_n \leqslant (|S|+1)^c$ for any integer $c>0$.

DEFINITION 1. Suppose R and S are as above, and $M=\sum_{i=1}^{t} Rx_i$ is a f.g. R-module. Define $M_n=\sum_{i=1}^{t} R_n x_i$ and $m_n=[M_n:F]$. Writing $X=\{x_1, ..., x_t\}$, define $\operatorname{rank}_{X;S}(M) = \overline{\lim}(m_n/d_n)$ and $\operatorname{rank}_S(M) = \inf\{\operatorname{rank}_{X;S}(M)\colon M \text{ is spanned by } X\}$.

This is *not* the usual definition of GK dimension of a module; in particular GK dim $R^{(2)}=1$, but rank $R^{(2)}=2$.

Note that $\operatorname{rank}_{X;S}(M) \leqslant |X|$, and thus $\operatorname{rank}_S(M) \leqslant$ the minimal number of elements that can span M. Also rank_S depends on the choice of generating set S for R; we shall discuss this later. First let us generalize the result of [1].

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LEMMA 2. *Suppose R has subexponential growth, and $m=\underline{\lim}(m_n/d_n)$. Then* $\operatorname{rank}_{X';S}(M)\geqslant m$ *for any spanning set X' of M.*

Proof. (inspired by [1]). Suppose $M=\sum_{i=1}^{t} Rx_i=\sum_{j=1}^{u} Rx'_j$. Write $x_i=\sum_{j=1}^{u} r_{ij}x'_j$; then there is some c for which $\{r_{ij}: 1\leqslant i\leqslant t,\ 1\leqslant j\leqslant u\}\subseteq R_c$, implying $x_i\in\sum_j R_c x'_j$. Let $X'=\{x'_1,\ldots,x'_u\}$, $M'_n=\sum_j R_n x'_j$, $m'_n=[M'_n:F]$, and $m'=\overline{\lim}(m'_n/d_n)=\operatorname{rank}_{X';S}(M)$. Then

$$M_n\subseteq\sum_j R_{n+c}x'_j=M'_{n+c}.$$

Given $\varepsilon>0$ there is n_0 such that $m'+\varepsilon>m'_n/d_n$ for all $n>n_0$, so $m_n\leqslant m'_{n+c}<(m'+\varepsilon)\,d_{n+c}$; furthermore, we can pick n_0 such that $m<m_n/d_n+\varepsilon$ for all $n>n_0$. Thus

$$m<(m'+\varepsilon)\,d_{n+c}/d_n+\varepsilon,$$

so

$$m/m'<d_{n+c}/d_n+(1+d_{n+c}/d_n)\,\varepsilon/m'\leqslant d_{n+c}/d_n+\varepsilon' m/m',$$

where $\varepsilon'=(1+(|S|+1)^c)\,\varepsilon/m$, implying

$$(1-\varepsilon')\,m/m'\leqslant d_{n+c}/d_n.$$

Iterating some number v times (applied to n, $n+c$, $n+2c$, etc.) yields

$$((1-\varepsilon')(m/m'))^v\leqslant d_{n+cv}/d_n;$$

taking logarithms yields

$$v(\log(1-\varepsilon')+\log(m/m'))\leqslant\log d_{n+cv}-\log d_n;$$

dividing by v and letting $v\to\infty$ yields

$$\log(1-\varepsilon')+\log(m/m')\leqslant 0-0=0$$

(the right side being 0 since R has subexponential growth). Letting $\varepsilon\to 0$ we see $\varepsilon'\to 0$ and thus $\log(1-\varepsilon')\to 0$, implying $\log(m/m')\leqslant 0$. Consequently $m\leqslant m'$. Q.E.D.

Inspired by this result let us define lower $\operatorname{rank}_{X;S}(M)=\underline{\lim}(m_n/d_n)$ and lower $\operatorname{rank}_S(M)=\sup\{\text{lower } \operatorname{rank}_{X;S}(M)\colon M \text{ is spanned by } X\}$.

COROLLARY 3. *If R has subexponential growth then* lower $\operatorname{rank}_S(M)\leqslant \operatorname{rank}_S(M)$.

Proof. The lemma shows lower $\operatorname{rank}_{X';S}(M)\leqslant\operatorname{rank}_S(M)$ for every spanning set X', so take the sup. Q.E.D.

THEOREM 4. *Fix a generating set S for R. If R has subexponential growth and the sequence $\{m_n/d_n\}$ converges to a limit m then*

$$\operatorname{rank}_S(M) = \text{lower } \operatorname{rank}_S(M) = m.$$

Proof. $m \leqslant \text{lower } \operatorname{rank}_S(M) \leqslant \operatorname{rank}_S(M) \leqslant m.$ Q.E.D.

COROLLARY 5. $\operatorname{rank}_S(R^{(t)}) = t$ *(for any generating set S of R), since taking the standard base we have each $m_n = td_n$, so that $m_n/d_n = t$, and certainly the constant sequence $\{t\}$ converges. Thus t is uniquely determined, and we have reproved* [1].

We would like to find other instances in which $\operatorname{rank}_S(M)$ can be determined from one particular spanning set of M. To this end we turn to rings of *polynomially bounded growth*, i.e., $d_n \leqslant n^t$ for some $t \in \mathbb{N}$ and all suitably large n. In this case one defines GK $\dim(R)$ to be $\overline{\lim} \log(d_n)/\log(n))$; GK stands for Gelfand–Kirillov. A standard reference for GK dimension is [2]; the basic facts are also to be found in [3]. It is well-known that the GK dimension is quite well-behaved when it also equals $\underline{\lim}(\log(d_n)/\log(n))$, which is called the *lower Gelfand–Kirillov dimension*, and that this equality holds in several interesting situations (affine PI-algebras, homomorphic images of enveloping algebras of finite dimensional Lie algebras, group algebras of nilpotent-by-finite groups, etc.). This leads us to the next fact.

LEMMA 6. *If* GK $\dim(R) =$ lower GK $\dim(R) < \infty$ *then* $\lim_{n\to\infty} \log(d_{cn})/\log(d_n) = 1$, *for any* $c > 1$.

Proof. Let $t = \text{GK} \dim(R)$. It is easy to see that if $t < 1$ then $t = 0$ and R is finite-dimensional, in which case d_n reaches $[R:F]$ for suitably large n, so that the assertion is trivial. Thus we may assume $t \geqslant 1$. Take n_0 large enough such that $|(\log d_n/\log n) - t| < t\varepsilon$ for all $n > n_0$. Then

$$t(1-\varepsilon) \leqslant \log d_n/\log n \leqslant t(1+\varepsilon),$$

implying

$$\begin{aligned}\log d_{nc}/\log d_n &= (\log d_{nc}/\log nc)(\log nc/\log n)/(\log d_n/\log n)\\ &\leqslant t(1+\varepsilon)(1+(\log c/\log n))/(t(1-\varepsilon)) \leqslant 1+3\varepsilon\end{aligned}$$

for large enough n.

Likewise $\log d_{nc}/\log d_n \geqslant t(1-\varepsilon)(1+(\log c/\log n)/(t(1+\varepsilon)) \geqslant 1-3\varepsilon.$

Q.E.D.

Lemma 6 clearly implies $\log d_{n+c}/\log d_n \to 1$ for any $c > 0$. We would like to conclude $d_{n+c}/d_n \to 1$, i.e., $\log d_{n+c} - \log d_n \to 0$ but this certainly does not follow formally: Let $d_n = 2^{2m+t}$ for $2^m < n \leqslant 2^{m+1}$. Then $\log d_n / \log n \to 2$ and $\log d_{2n} / \log d_n = (2m+2+t)/(2m+t) \to 1$, but d_{n+2}/d_n takes on the value 4 whenever n is a power of 2. Thus we must define a new condition.

DEFINITION 7. R has *strictly defined* growth (with respect to S) if $\overline{\lim}_{n \to \infty} (d_{n+1}/d_n) = 1$.

Remark 8. If R has strictly defined growth then $\overline{\lim}_{n \to \infty} (d_{n+c}/d_n) = 1$ for any natural number c.

THEOREM 9. *If R has strictly defined growth then* $\operatorname{rank}_S(M) = \operatorname{rank}_{X';S}(M)$ *for every spanning set X' of M.*

Proof. Suppose $M = \sum_{i=1}^{t} Rx_i = \sum_{j=1}^{u} Rx'_j$. Write $x'_j = \sum_{i=1}^{t} r_{ij} x_i$; then there is some c for which $\{r_{ij} : 1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant u\} \subseteq R_c$, implying $x'_j \in \sum_i R_c x_i$. Let $M'_n = \sum_j R_n x'_j$, $m'_n = [M'_n : F]$, and $m' = \overline{\lim}_{n \to \infty} (m'_n / d_n)$. As in Lemma 2 define $m = \overline{\lim}(m_n/d_n)$. We want to prove $m = m'$. Pick any $\varepsilon > 0$. Note

$$M'_n \subseteq \sum_i R_{n+c} x_i = M_{n+c'}$$

implying there is n_0 such that $m'_n \leqslant m_{n+c} \leqslant (m+\varepsilon)\, d_{n+c}$ for all $n \geqslant n_0$; we can choose n_0 so that $\log d_{n+c} \leqslant \varepsilon + \log d_n$ for all $n \geqslant n_0$. Then

$$\log m'_n \leqslant \log(m+\varepsilon) + \varepsilon + \log d_n,$$

so

$$\log(m'_n/d_n) = \log m'_n - \log d_n \leqslant \log(m+\varepsilon) + \varepsilon.$$

Taking limits and putting $\varepsilon \to 0$ yields $\log m' \leqslant \log m$, so $m' \leqslant m$. By symmetry, $m \leqslant m'$. Q.E.D.

Note on Proof. The definition of $\operatorname{rank}_S(M)$ still depends on the choice of S. Thus each choice of S could conceivably give us a different rank, but the rank of $R^{(t)}$ will always be t, as noted above.

EXAMPLE 10. The following algebras have strictly defined growth:

(i) If R is the polynomial algebra $F[\lambda_1, ..., \lambda_m]$ then as in [3, Section 6.2] we have (for $S = \{\lambda_1, ..., \lambda_t\}$)

$$d_{n+1}/d_n = \binom{m+n+1}{n+1} \Big/ \binom{m+n}{n} = \frac{m+n+1}{n+1} \to 1.$$

(ii) More generally if $R=T[\lambda]$ and T has strictly defined growth with respect to S then R has strictly defined growth with respect to $\{\lambda\}\cup S$.

Proof. $R_n=\sum T_{n-i}\lambda^i$, so $d_n=\sum d_{n-i}(T)$, where $d_k(T)=[T_k:F]$. Thus $d_{n+1}/d_n=1+d_{n+1}(T)/\sum_{i\leqslant n} d_{n-i}(T)$. Given $\varepsilon>0$ we take k large enough such that $d_n(T)>(1-\varepsilon)\,d_{n+1}(T)$ for all $n\geqslant k$. Then for all $i\geqslant k$,

$$d_i(T)>(1-\varepsilon)^{n+1-i}\,d_{n+1}(T),$$

implying

$$\sum_{k\leqslant i\leqslant n} d_i(T)>d_{n+1}(T)\sum_{k\leqslant i\leqslant n}(1-\varepsilon)^{n+1-i}$$
$$=(1-\varepsilon)\,d_{n+1}(T)(1-(1-\varepsilon)^{n+1-k})/\varepsilon.$$

Take n large enough such that $(1-\varepsilon)^{n+1-k}<1/2$. Then

$$\sum_{k\leqslant i\leqslant n} d_i(T)>(1-\varepsilon)\,d_{n+1}(T)/2\varepsilon,$$

$$\text{so}\quad d_{n+1}(T)/\sum_{k\leqslant i\leqslant n} d_i(T)<2\varepsilon/(1-\varepsilon)$$

Thus for any ε we have $d_{n+1}/d_n<1+2\varepsilon/(1-\varepsilon)$ for all suitably large n. This clearly implies $d_{n+1}/d_n\to 1$ as $n\to\infty$.

(iii) If the associated graded algebra of an affine algebra with filtration R has strictly defined growth, then R also has strictly defined growth, since the growth functions "match." In particular, enveloping algebras of finite dimensional Lie algebras have strictly defined growth. Indeed the associated graded algebra of an enveloping algebra is a polynomial algebra over a field, and thus has strictly defined growth, by (i) or (ii).

Note that strictly defined growth does not formally imply polynomial growth, e.g., if $\log d_n\approx(\log n)^2$ then $\log d_{n+1}-\log d_n\approx(\log(n+1))^2-(\log n)^2=(\log(n+1)-\log n)(\log(n+1)+\log n)\approx\log(1+1/n)(2\log n)\approx(2/n)\log n\to 0$.

This rank has several nice properties.

PROPOSITION 11. *Assume S is a given generating set for R. Then rank_S has the following properties*:

(i) $\operatorname{rank}_S(R)=1$ *if R has subexponential growth, by Theorem* 4.

(ii) *If M is a homomorphic image of N then* $\operatorname{rank}_S(M)\leqslant\operatorname{rank}_S(N)$.

(iii) *If $M\leqslant N$ then* $\operatorname{rank}_S(N)\leqslant\operatorname{rank}_S(M)+\operatorname{rank}_S(N/M)$.

(iv) $\operatorname{rank}_S(M_1 \oplus M_2) \leqslant \operatorname{rank}_S(M_1) + \operatorname{rank}_S(M_2)$, *equality holding when R has strictly defined growth.*

Proof. Just take the correct spanning sets; in (iii) take a spanning set of N which consists of a spanning set of M and a set of representatives of a spanning set of N/M.

COROLLARY 12. *When R has strictly defined growth with respect to S, rank_S is a Sylvester module rank function in the sense of* [4, p. 97].

Unfortunately it is not clear whether this takes values in $(1/n)\mathbb{Z}$.

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